

REVIEW OF ASYMPTOTES

SUMMARY

Let $f(x) = P(x)/D(x)$, where $P(x)$ and $D(x)$ have no common factors. Then

1. the *vertical line* given by $x = b$ is an asymptote if $D(b) = 0$;
2. the *horizontal line* given by $y = k$ is an asymptote if

$$\lim_{|x| \rightarrow \infty} f(x) = k;$$

3. the *slant line* given by $y = mx + b$ is an asymptote if the degree of $P(x)$ is one more than the degree of $D(x)$ and

$$\frac{P(x)}{D(x)} = mx + b + \frac{R(x)}{D(x)}.$$

(Use synthetic division if $D(x)$ is of the form $x-b$; otherwise, use long division.)

GRAPHING RATIONAL FUNCTIONS

We now know a great deal about the asymptotes of certain rational functions but how can we put this information to use in order to graph the function? Generally, we follow the procedure summarized below.

1. Find the graph and asymptotes, if any.
2. Find and plot any points where the graph passes through an asymptote or the coordinate axes.
3. The plane is now divided into one or more regions with asymptotes as boundaries. Plot selected points within each region to determine the shape of the graph within that region. Remember to make use of the fact that the boundaries of these regions are asymptotes and that the distance between the curve and the asymptote must decrease as you move out from the origin.

Examples 1-3 illustrate this procedure.

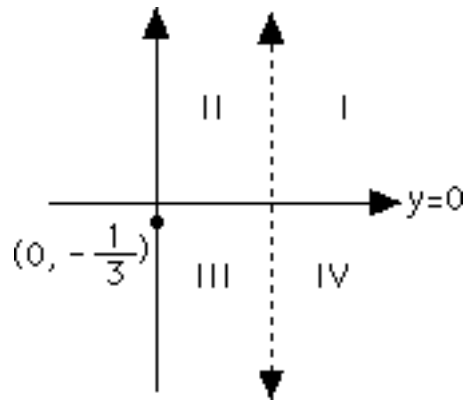
EXAMPLES: Graph each rational function.

1. Let the function f be defined by the equation

$$y = \frac{1}{x - 3}.$$

Step 1. The asymptotes are given by $x = 3$ and $y = 0$ (from Example 1).

Step 2. The curve passes through $(0, -\frac{1}{3})$ and does not pass through any asymptotes since $1/(x - 3) = 0$ has an empty solution set. The asymptotes and this point are shown below. Also notice that the only point of discontinuity



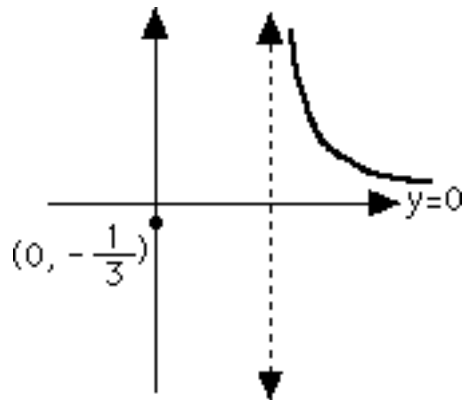
is $x = 3$. The plane is divided into Regions I, II, III, and IV.

Step 3. Region I

x	y
3.5	2
4	1
5	$\frac{1}{2}$
6	$\frac{1}{3}$

Plot the points corresponding to $x = 3.5, 4, 5,$ and 6 . Draw the curve in the region, as shown in Figure 4.9. Use the facts that the lines given by $x = 3$ and $y = 0$ are asymptotes and that the graph of the function doesn't cross these lines.

Region II: since f is a function and its graph does not cross the asymptote given by $y = 0$, and since there is a point $(0, -\frac{1}{3})$ below this line, there are no points in Region II.



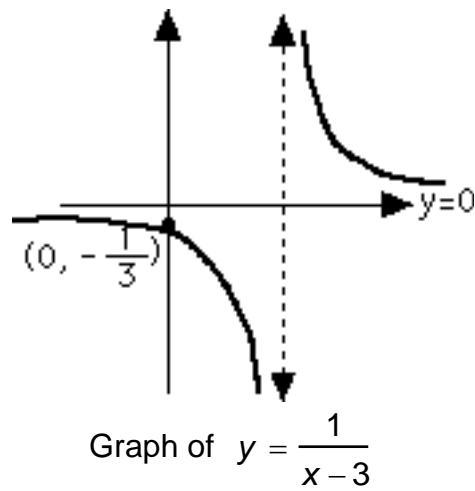
Part of the graph of $y = \frac{1}{x-3}$

Notice that f is discontinuous at $x = 3$.

Region III

x	y
-1	$-\frac{1}{4}$
1	$-\frac{1}{2}$
2	-1
2.5	-2

Draw the curve in this region, as shown below. Use the



facts that the lines given by $x = 3$ and $y = 0$ are asymptotes and that the curve doesn't cross these lines.

Region IV: since the graph does not cross the asymptote given by $y = 0$, there are no points in Region IV. The graph is complete as shown in the figure above.

2. Let f be a function defined by the equation

$$y = \frac{2x^2 - 3x + 5}{x^2 - x - 2}.$$

For the intercepts, let

$$x = 0: y = \frac{2 \cdot 0^2 - 3 \cdot 0 + 5}{0^2 - 0 - 2} = -\frac{5}{2}$$

and

$$y = 0: 0 = \frac{2x^2 - 3x + 5}{x^2 - x - 2}$$
$$2x^2 - 3x + 5 = 0$$

No real roots since $b^2 - 4ac = 9 - 4 \cdot 2 \cdot 5 < 0$.

For the asymptotes see pages 180 and 182.

Note: the order in which you plot points in these regions doesn't matter.

Step 1. The asymptotes are given by $x = 2$, $x = -1$, and $y = 2$ (from Example 2).

Step 2. The curve passes through $(0, -5/2)$ and also passes through the horizontal asymptote at $(9, 2)$. Label Regions I, II, III, IV, V and VI as shown in Figure 4.11. The only points of discontinuity are at the vertical asymptotes, namely $x = -1$ and $x = 2$.

Step 3. Plot points; do each region separately.

Region I

x	y
3	3.5
4	2.5
9	2

← crosses asymptote at this point and crosses into Region VI

Region VI

x	y
9	2
12	1.98

← crosses asymptote

Region II is empty since f is a function and its graph does not cross $y = 2$ between $x = -1$ and $x = 2$.

x	y
-2	4.75
-4	2.72

x	y
1	-2
1.5	-4
0.5	-1.8
0	$-\frac{5}{2}$

← y-intercept

